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Fixed Point Theorem in Generalized Metric Spaces Using Quasi-Contractive Type Maps

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ABSTRACT

In this paper, we state and prove a fixed point theorem in generalized metric space by using a quasi-contractive map. Result presented in this paper generalize and extend results of Banach [1], Kannan [4], Nadler [10], Reich [11], etc. in the setting of generalized metric spaces.

I. PRELIMINARIES AND MAIN RESULT

In 1992, Dhage [3] introduce a new class of generalized metric called as *D*-metric. Dhage claimed in his doctoral research work that *D*-metric provides a generalization of ordinary metric functions. In 2006, Mustafa and Sims [8] introduced the concept of *G*-metric spaces to overcome fundamental flaws in Dhage's theory of generalized metric spaces as follows:

Definition 1.1. Let X be a non-empty set, and let $G: X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following axioms: for all $x, y, z, a \in X$,

($G1$) $G(x, y, z) = 0$ if $x = y = z$;

($G2$) $G(x, x, y) > 0$ with $x \neq y$;

($G3$) $G(x, x, y) \leq G(x, y, z)$ with $z \neq y$;

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$;

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$;

then the function G is called a **Generalized metric** or more specifically a ***G*-metric** on X and the pair (X, G) is called a ***G*-metric space**.

Definition 1.2.[8] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be ***G*-convergent** to a point $x \in X$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, i.e., for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 1.3.[8] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be a ***G*-Cauchy sequence** if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_k) < \varepsilon$, for all $n, m, k \geq N$; i.e., if $G(x_n, x_m, x_k) \to 0$ as $n, m, k \to \infty$.

Definition 1.4.[8] A G-metric space (X, G) is said to be ***G*-complete** if every G-Cauchy sequence in (X, G) is G-convergent in X.

Definition 1.5.[8] A G – metric space (X, G) is called a **symmetric *G*-metric space** if

$$G(x, y, y) = G(x, x, y), \text{ for all } x, y \in X.$$

Motivated by the work of Mustafa and Sims [8], various researchers (see, e.g., [5-7],[9]) have proved number of well known results in G-metric spaces.

Now, we introduce quasi-contraction mappings in G-metric spaces as follows:

Definition 1.6. A mapping $T : X \to X$ of a G-metric space X into itself is said to be quasi- contraction iff there exists a number q, $0 \le q < 1$ such that

$G(Tx, Ty, Tz) \le q \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$.

Definition 1.7.[2] Let T be a mapping of G-metric space X into itself. For $A \subset X$, define

(i) $\delta(A) = Sup\{G(a,b,c) : a,b,c \in A\}$ and

(ii) for each $x \in X$,

$O(x,n) = \{x, Tx, T^2x, T^3x, \ldots T^nx\}, n = 1,2,3..$ and $O(x,\infty) = \{x, Tx, T^2x, \ldots\}$.

A space (X, G) is said to be T-orbitally complete iff every Cauchy sequence which is contained in $O(x,\infty)$ for some $x \in X$ converges in X.

Theorem 1.1. Let (X, G) be a G-metric space. Suppose that $T : X \to X$ is a quasi- contraction and X is T-orbitally complete. Then we have

(i) T has a unique fixed point.

(ii) $\lim_n T^nx = z$ for all $x \in X$.

(iii) $G(T^nx, z, z) = \dfrac{q^n}{1-q} G(x, Tx, Tx)$ for all $x \in X$ and $n \in N$.

Proof: For each $x \in X$ and $0 \le i \le n$, $0 \le j \le n$, we have

$$\begin{aligned}
G(T^ix, T^jx, T^jx) &= G(TT^{i-1}x, TT^{j-1}x, TT^{j-1}x) \\
&\le q \max\left\{\begin{matrix} G(T^{i-1}x, T^{j-1}x, T^{j-1}x), G(T^{i-1}x, TT^{i-1}x, TT^{i-1}x), \\ G(T^{j-1}x, TT^{j-1}x, TT^{j-1}x), G(T^{j-1}x, TT^{j-1}x, TT^{j-1}x) \end{matrix}\right\} \\
&= q \max\left\{\begin{matrix} G(T^{i-1}x, T^{j-1}x, T^{j-1}x), G(T^{i-1}x, T^ix, T^ix), \\ G(T^{j-1}x, T^jx, T^jx), G(T^{j-1}x, T^jx, T^jx) \end{matrix}\right\} \\
&\le q\delta\left[O_T(x,n)\right]
\end{aligned}$$

where

$$\delta\left[O_T(x,n)\right] = \max\{G(T^ix, T^jx, T^jx) : 0 \le i, j \le n\}.$$

Since $0 \le q < 1$, there exists $h_n(x) \le n$ such that

$G(x, T^{h_n(x)}x, T^{h_n(x)}x) = \delta\left[O_T(x,n)\right]$.

Then we have

$$\begin{aligned}
G(x, T^{h_n(x)}x, T^{h_n(x)}x) &\le G(x, Tx, Tx) + G(Tx, T^{h_n(x)}x, T^{h_n(x)}x) \\
&\le G(x, Tx, Tx) + q\delta(O_T(x,n)) \\
&= G(x, Tx, Tx) + qG(x, T^{h_n(x)}x, T^{h_n(x)}x).
\end{aligned}$$

It implies that

$$G(x, T^{h_n(x)}x, T^{h_n(x)}x) \le \frac{1}{1-q} G(x, Tx, Tx) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

For all n, $m \le 1$ and $n < m$, it follows from the quasi contractive condition of T and (1) that

$$\begin{aligned}
G(T^nx, T^mx, T^mx) &= G(TT^{n-1}x, T^{m-n+1}T^{n-1}x, T^{m-n+1}T^{n-1}x) \\
&\le q\delta(O_T(T^{n-1}x, m-n+1))
\end{aligned}$$

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$$
\begin{aligned}
&= qG(T^{n-1}x, T^{m-n+1}T^{n-1}x, T^{m-n+1}T^{n-1}x) \\
&= qG(TT^{n-2}x, T^{m-n+2}T^{n-2}x, T^{m-n+2}T^{n-2}x) \\
&\le q^2\delta(O_T(T^{n-2}x, m-n+2) \\
&\le \ldots \\
&\le q^n\delta\left[O_T(x,m)\right] \\
&\le \frac{q^n}{1-q}G(x,Tx,Tx)\ldots\ldots(\mathrm{A})
\end{aligned}
$$

This gives $\{T^n x\}$ is a Cauchy sequence in X. Since X is T-orbitally complete, there exists z belongs to X such that

$$\lim_{x\to\infty} T^n x = z\ldots.(2)$$

By using the quasi contractive condition, we get

$$
\begin{aligned}
G(z,Tz,Tz) &\le G(z,T^{n+1}z,T^{n+1}z) + G(T^{n+1}z,Tz,Tz) \\
&= G(z,T^{n+1}z,T^{n+1}z) + G(TT^{n}z,Tz,Tz) \\
&\le G(z,T^{n+1}z,T^{n+1}z) + q.\max\begin{Bmatrix} G(T^n z,z,z), \\ G(T^n z, TT^n z, TT^n z), \\ G(z,Tz,Tz), \\ G(z,Tz,Tz) \end{Bmatrix} \\
&= G(z,T^{n+1}z,T^{n+1}z) + q.\max\begin{Bmatrix} G(T^n z,z,z), \\ G(T^n z, T^{n+1} z, T^{n+1} z), \\ G(z,Tz,Tz), \\ G(z,Tz,Tz) \end{Bmatrix}.
\end{aligned}
$$

................(3)

Taking limit as n tends to infinity in (3) and using (2), we get

$$G(z,Tz,Tz) \le qG(z,Tz,Tz).$$

Since $0 \le q < 1$, we obtain

$$G(z,Tz,Tz) = 0.$$

This gives, T has a fixed point z in X.

To prove uniqueness of fixed point, let w be another fixed point of T. Then by using quasi-contractive condition on T, we have

$$G(z,w,w) = G(Tz,Tw,Tw) \le q\max\begin{Bmatrix} G(z,w,w), G(z,Tz,Tz), \\ G(w,Tw,Tw), G(w,Tw,Tw), \end{Bmatrix}$$

$$G(z,w,w) \le qG(z,w,w)$$

a contradiction, hence $z = w$. This proves uniqueness of fixed point.

Also, by taking limit as $n \to \infty$ in (A), we have

$$G(T^n x, z, z) = \frac{q^n}{1-q} G(x, Tx, Tx) .$$

Hence result follows.

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