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Generalization of Kaplansky Theorem

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ABSTRACT. Let R be a ring with unity and satisfies a condition $[x^m, y^n] = 0$, for all $x, y \varepsilon R$. In this paper, we extend a well known result proved by Kaplansky.

KEYWORDS: Torsion free ring, Commutator, power map

I. INTRODUCTION

Let R be a ring with unity. An element x of a ring R is said to be n-torsion free if $nx = 0$ implies $x = 0$. If $nx = 0$ implies $x = 0$, for every $x \varepsilon R$, then we say that R is n-torsion free. A ring is said to be commutative if and only if $[x, y] = 0$, for every pair x, y of ring elements. This definition of commutativity of ring prompts us to investigate the commutativity of a ring if there exists a positive integer n larger than1 such that $[x^n, y] = 0$, for all pairs x, y of the ring elements. The non commutative ring of 3×3 strictly upper triangular matrices over the ring Z of integers rules out the possibility of arbitrary rings with $[x^n, y] = 0$ to be commutative. Despite such bad examples, algebraists have been investigating the classes of rings which turn out to be commutative under the mentioned condition.

In this direction, Kaplansky [5] proved that a semisimple ring R in which there exists a positive integer $n \geq 1$ such that $[x^m, y^n] = 0$, for all $x, y \varepsilon R$ must be commutative. This result attracted many algebraists including Carl Faith [3] and I. N. Herstein [4]. However most of the results available in the literature are about very restricted classes of rings. For example Faith [1] established the result for division ring whereas Herstein [2] proved commutativity of rings in which commutator ideal is not nil. In this paper, we extend the result for ring with unity 1, imposing torsion condition on the elements of the ring.

II. MAIN RESULT

We begin with the following lemma which is required to prove our theorem.

LEMMA 2.1[8, Lemma1]. Suppose R is an associative ring with unity. For any $x \varepsilon$ R, let

$$S_0^r = x^r$$

and

$$S_k^r = S_{k-1}^r(1+x) - S_{k-1}^r(x),\ k \geq 1.$$

Then

(i) $S_{r-1}^r(x) = !r\,[\frac{1}{2}(r\text{-}1) + x]$

(ii) $S_r^r(x) = !r$

(iii) $S_j^r(x) = 0$, for j $\geq r$.

THEOREM. Let R be a ring with unity in which there exists a pair of positive integers $m \geq 1$, $n \geq 1$ such that $[x^m, y^n] = 0$, for all $x, y \varepsilon R$. If R is $!m!n$ torsion free, then R is necessarily commutative.

PROOF. Using the notations of the above lemma, the condition of our theorem can be written as follows:

$[S_0^m(x), y^n] = 0$, for all $x, y \varepsilon R$.

On replacing x by $1+x$ in the above identity, we have

$$[S_0^m(1+x), y^n] = 0, \text{ for all } x, y \varepsilon R \qquad (1)$$

That is,

[$S_1^m(x) + S_0^m(x), y^n$] = 0, for all $x, y \in R$.

Since commutator function is linear in both the coordinates, we have

[$S_1^m(x), y^n$] + [$S_0^m(x), y^n$] = 0, for all $x, y \in R$.

In view of (1), this yields

[$S_1^m(x), y^n$] = 0, for all $x, y \in R$.

Again replace x by $1+x1$ to get

[$S_1^m(1+x), y^n$] = 0, for all $x, y \in R$.

Now repeating the process (m-1) times and using the Lemma 2.1, we obtain

[$S_{m-1}^m(x), y^n$] = 0, for all $x, y \in R$.

Thus by (i) of Lemma 2.1, we get

$$[\tfrac{1}{2}(m\text{-}1)\ !m + !m\ x,\ y^n] = 0.$$

i.e,

$$[\ !m\ x,\ y^n] = 0, \text{ for all } x, y \in R.$$

Now once again writing the above relation in the notation of Lemma 2.1, we have

$$!m\ [\ x,\ S_0^n(y)] = 0, \text{ for all } x, y \in R.$$

This time working in the second coordinate of the commutator and proceeding in this way as above, we finally get

$!m!n$ [x, y] = 0, for all $x, y \in R$.

Since R is $!m!n$ torsion free, we have [x, y] = 0, for all $x, y \in R$ which shows that R is commutative.

Remark2.1. Evidently, for m = 1 or n = 1, the above theorem turns out to be an extension of Kaplansky theorem. Our theorem also includes the theorem of Faith,Lithman and many others.

REMARK2.2. A cursory look at the proof of the theorem will reveal that the result remains still valid if the ring under consideration is m! as well as n! torsion free Also the condition of the hypothesis can be further weakened by assuming that the commutator in R are m! and n! torsion free.

REMARK2.3. The following example demonstrates that the torsion condition on the commutators of the ring of our theorem can not be dropped.

EXAMPLE. Consider the ring

$$R = \left\{ \begin{array}{c} aI_3 + D_0 = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix},\ I_3\ is\ 3 \times 3 \\ \text{identity matrix and } a, b, c, d \in GF(2) \end{array} \right\}.$$

It is readily verify that R is a noncommutative ring with unity satisfying [x^2, y] = 0, for all x, $y \in$ R. Indeed, R is not 2-torsion free.

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